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Mh4714 Week 7

Week 7

The following example shows one familiar application of the Intermediate Value Theorem.

Example 0.1

Use the IMVT to prove that the equation $x^3 - 3x^2 + 10x - 4 = 0$ has a solution between 0 and 1.
If we let $f(x) = x^3 - 3x^2 + 10x - 4$ then we know that $f(x)$ is continuous at every $x \in \mathbb{R}$ and so is continuous over [0,1].
We have $f(0) = -4$ and $f(1) = 4$ and so by the IMVT we conclude that there is some $c \in (0, 1)$ such that $f(c) = 0$.

Recall that some quadratics have no real roots.

Example 0.2

- $x^2 + 1$ has no real roots since

$$x^2 + 1 = 0 \Rightarrow x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm i.$$

- $x^2 + x + 1$ has no real roots since

$$x^2 + x + 1 = 0 \Rightarrow x = \frac{-1 \pm \sqrt{(-1)^2 - 4(1)(1)}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1}{2} \pm \frac{3}{2}i.$$

By contrast we can show that any polynomial of odd degree has at least one real root. The following example illustrates the general argument:

Example 0.3

Consider the polynomial $f(x) = x^3 - 10x^2 + 7x + 1$:

$$x^3 - 10x^2 + 7x + 1 = x^3 \left(1 - \frac{10}{x} + \frac{7}{x^2} + \frac{1}{x^3}\right)$$

Note that

$$\lim_{x\to\infty} \left(1 - \frac{10}{x} + \frac{7}{x^2} + \frac{1}{x^3}\right) = 1$$

and

$$\lim_{x\to-\infty} \left(1 - \frac{10}{x} + \frac{7}{x^2} + \frac{1}{x^3}\right) = 1.$$

This means that $1 - \frac{10}{x} + \frac{7}{x^2} + \frac{1}{x^3}$ becomes arbitrarily close to 1 as x approaches ∞ and as x approaches $-\infty$.
It follows that there is some $a < 0$ such that

$$1 - \frac{10}{x} + \frac{7}{x^2} + \frac{1}{x^3} > 0 \text{ for all } x \leq a.$$

and there is some $b > 0$ such that

$$1 - \frac{10}{x} + \frac{7}{x^2} + \frac{1}{x^3} > 0 \text{ for all } x \geq b.$$

Since $a^3 < 0$ and $b^3 > 0$ we have

$$f(a) = (\text{-ve})(+\text{ve}) < 0$$

and

$$f(b) = (+\text{ve})(+\text{ve}) > 0$$

And so we have a continuous function $f(x)$ with $f(a) < 0 < f(b)$ and so, by the IMVT, it follows that there is some point $c \in (a, b)$ such that $f(c) = 0$. That is, there is some $x \in \mathbb{R}$ such $x^3 - 10x^2 + 7x + 1 = 0$.

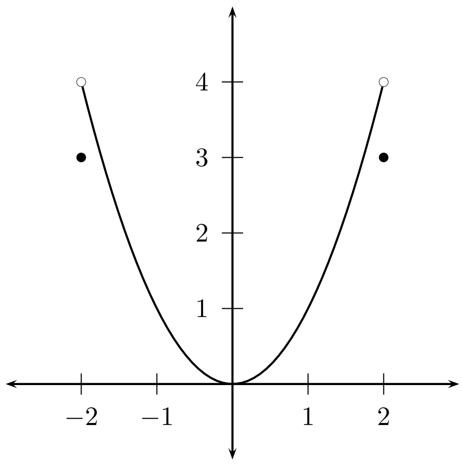
0.0.1 Boundedness Properties of Continous Functions

Theorem 0.4

Let f be a real-valued funtion. If f is continuous over a closed bounded interval $[a, b]$ then f has a maximum and a minimum value in $[a, b]$.

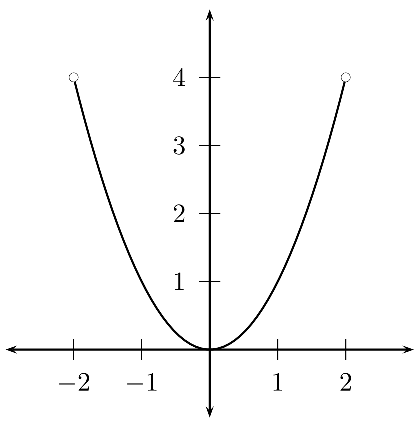
We will not supply a proof of this result but rather look at some counter-examples.

- $f(x) = \begin{cases} x^2, x \in (-2,2), \\ 3, x = \pm 2. \end{cases}$



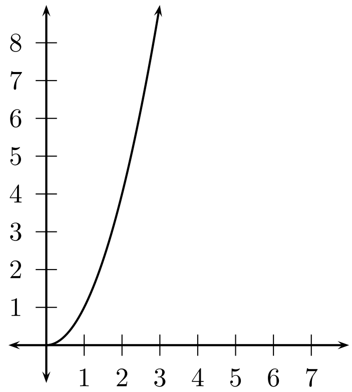
is an example of a function *not continuous* over a closed bounded interval. f has no maximum value in the interval.

- $f(x) = x^2, x \in (-2,2)$



is a continuous function defined over a bounded *not closed* interval. f has no maximum value in the interval.

- $f(x) = x^2, x \in [0, \infty)$



is a continuous function defined over an *unbounded* closed interval. f has no maximum value in the interval.

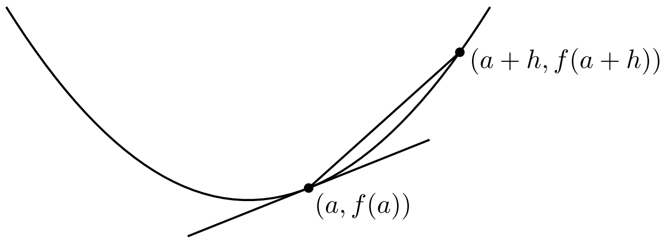
0.1 Differentiation

Definition 0.5

Let f be a real-valued function defined over an interval containing $a \in \mathbb{R}$.
If: $\displaystyle\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ exists then f is said to be *differentiable* at a.
The value of $\displaystyle\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ is called the *derivative* of f at a and is denoted usually as $f'(a)$ and sometimes as $\displaystyle\left.\frac{df(x)}{dx}\right|_{x=a}$.

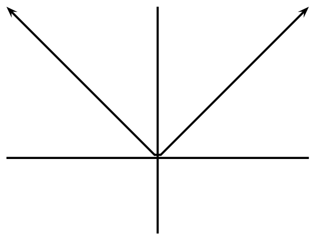
Notes:
The limit $\displaystyle\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ can also be written as $\displaystyle\lim_{x\to a} \frac{f(x) - f(a)}{x-a}$
The expression $\displaystyle\frac{f(a+h) - f(a)}{h}$ or $\displaystyle\frac{f(x) - f(a)}{x-a}$ is known as a *Newton Quotient*.

The following diagram shows that the Newton Quotient is the slope of the chord joining the points $(a, f(a))$ and $(a+h, f(a+h))$.
As h approaches 0 the slope of the chord approaches the slope of the tangent to the curve at $(a, f(a))$.



From this observation we arrive at the fact that $f'(a)$ is the slope of the tangent to the curve $y = f(x)$ at the point $f(a)$ and is thus also known as the slope of the curve.

We can infer from this that a function f is differentiable at a if there is a well defined tangent at the point $(a, f(a))$.
Consider, for example, the function $f(x) = |x|$. This has graph:



The Newton Quotient at 0 is

$$\frac{f(0+h) - f(0)}{h} = \frac{|0+h| - |0|}{h} = \frac{|h|}{h}$$

In order to examine $\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ we must use left-hand and right-hand limits:

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1$$

and

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

And we see that $\lim_{h \to 0} \frac{|0+h| - |0|}{h}$ does not exist. Therefore $|x|$ is not differentiable at 0.

There is a connection between differentiability and continuity.

Theorem 0.6

If f is differentiable at $a \in \mathbb{R}$ then f is continuous at a.

Proof

$$\lim_{x\to a} f(x) = \lim_{x\to a}(f(x) - f(a) + f(a)) = \lim_{x\to a}\left(\frac{f(x)-f(a)}{x-a}(x-a)\right) + f(a)$$
$$= \lim_{x\to a}\frac{f(x)-f(a)}{x-a}\lim_{x\to a}(x-a) + f(a) = f'(a).0 + f(a) = f(a).$$

That is, $\lim_{x\to a} f(x) = f(a)$

That is, f is continous at a. $\square$

The above theorem proves that differentiability implies continuity. However, the converse is not true, that is, a function can be continuous but not differentiable at a point.

Example 0.7

The function $f(x) = |x|$ is continuous at 0 because

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$$
$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} -x = 0$$

The left-hand and right-hand limits both exist and are both equal to 0 and so it follows that $\lim_{x\to 0} f(x) = 0 = f(0)$, therefore f is continuous at 0. We have already shown the $|x|$ is not differentiable at 0.

Note: If f is differentiable at every point of (a, b) then f is said to be *differentiable over* (a, b).

0.1.0.1 Some properties of derivatives..
If $u(x)$ and $v(x)$ are both differentiable at x then

1. uv is differentiable at x with

$$(uv)' = u'v + uv'$$

2. If $v(x) \neq 0$ then $\dfrac{u}{v}$ is differentiable at x with

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

3. If u is differentiable at $v(x)$ then

$$(u(v(x)))' = u'(v(x))v'(x).$$

   This is known as the *Chain Rule*.